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Quadratic relations in continuous and discrete Painlevé equations

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Abstract. The quadratic relations between the solutions of a Painlevé equation and that of a different one, or the same one with a different set of parameters, are investigated in the continuous and discrete cases. We show that the quadratic relations existing for the continuous P_{II}, P_{III}, P_V and P_{VI} have analogues as well as consequences in the discrete case. Moreover, the discrete Painlevé equations have quadratic relations of their own without any reference to the continuous case.

1. Introduction

Painlevé equations ($\mathbb{P}$'s) [1], being the prototypes of integrable ordinary differential equations, are characterized by a wealth of special properties which makes their study both fascinating and challenging. Among these properties a special role is played by exact relations which can relate either the solutions of two different $\mathbb{P}$'s, or the solutions of the same Painlevé equation for different sets of parameters [2]. These relations are known as Miura and auto-Bäcklund/Schlesinger transformations, respectively [3]. These transformations are given in the form of rational expressions involving the solution of the Painlevé equation as well as its derivative. Another class of relations is the one due to the point symmetries of the equations. They typically assume a local form i.e. a relation where the derivative does not enter at all.

While the above class of relations is well known and thoroughly studied [4] (at least in the case of continuous $\mathbb{P}$'s) there exists a class of relations which has not attracted much interest. These relations are of a special type in the sense that they relate the solution of a given Painlevé equation to the *square* of that of some other $\mathbb{P}$ (which can be the same as the initial one). The fact that such quadratic relations exist can be traced back to the fact that the Painlevé equations have singularities which, in general, can be simple or double poles (or zeros). Thus a quadratic relation relates a solution which has only simple poles to one where all poles are double. These relations generally exist only for special values of the parameters of a given Painlevé equation. Let us make these considerations clearer through an example. Consider the P_{III} equation [5]:

$$w'' = \frac{w'^2}{w} - \frac{w'}{t} + \frac{1}{t}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w}. \tag{1.1}$$

When $\gamma\delta \neq 0$ one can always, through a change of variables which are both dependent and independent, scale γ to 1, δ to -1, and α and β are two genuine parameters. Moreover,

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Schlesinger transformations allow us to go from one set (α, β) to some other set, so zero values for these two parameters are not really special. (A caveat is mandatory at this point. The Schlesinger transformations do not allow one to generate arbitrary values of the parameters but only values which differ by some integer. Thus when we start from zero values for α and β we can transform them to values in $\mathbb{Z}$ but not to values in $\mathbb{R}$. Still whenever this is possible we consider that the parameters α and β are indeed present and thus the equation is essentially in generic form.) If only one of the quantities γ or δ vanishes, there is one genuine parameter left. However, if both γ and δ vanish we obtain an equation with no parameter at all, since then both α and β can be scaled to 1 and -1, respectively, for instance (unless one or both vanish, in which cases (1.1) is no longer a transcendental equation). So the equation

$$w'' = \frac{w'^2}{w} - \frac{w'}{t} + \frac{1}{t}(w^2 - 1) \tag{1.2}$$

where no Schlesinger transformations exist, that could regenerate γ or δ, is really a zero-parameter transcendental equation in its own right, which we call the zero-parameter P_{III}. (The term 'zero parameter' is to be understood in the light of the remark above, in the sense that there is no way to introduce, through Schlesinger transformations, a parameter even taking values in $\mathbb{Z}$.) If in this equation we set $w = u^2$, $t = s^2/2$ we obtain for x as a function of s exactly equation (1.1) with $\gamma = 1$, $\delta = -1$ and zero values for α and β. This shows that a quadratic relation transforms the zero-parameter P_{III} into a particular case of the full P_{III} (non-zero values of α and β can be regenerated through Schlesinger transformations). As we shall see in what follows, this is not the only known instance of such a quadratic relation. When some constraints on the parameters are satisfied, the P_V equation is related to P_{III} through such a quadratic relation, and also to a reduced case of itself, while P_{VI} can be related to itself.

What do these results imply for the discrete Painlevé equations (d-$\mathbb{P}$) [6]? Since the Painlevé equations possess discrete forms, it is expected that the quadratic relations carry over from the continuous systems to their discrete analogues. Moreover, as we have explained in previous work, some of the discrete (difference) Painlevé equations are contiguity relations of the continuous ones [7]. Thus, a quadratic relation between solutions of the continuous $\mathbb{P}$'s must have visible consequences in the discrete case. Lastly, as the d-$\mathbb{P}$s are, in some sense, richer than the continuous ones, they may possess some quadratic relations of their own, without reference whatsoever to continuous systems. Let us illustrate this last point with a particular case of the d-P_I equation:

$$\overline{x} + \underline{x} + x = \frac{z}{x} \tag{1.3}$$

where $z = \alpha n + \beta$ for some constant α, β while x denotes x_n and $\overline{x} \equiv x_{n+1}$, $\underline{x} \equiv x_{n-1}$. The right-hand side of the standard d-P_I is written as $t + z/x$ where t is a constant that, when it is not zero, is usually scaled to 1. Here we consider the $t = 0$ case and the corresponding equation, (1.3), was dubbed in [8] d-P_0 and has no non-trivial continuous limit. We multiply both sides of (1.3) by x and introduce the variables $X = x^2$ and $y = x\overline{x}$. We have thus from (1.3), $y + \underline{y} + X = z$ and, from the definition of y, $X\overline{X} = y^2$. Eliminating X between the two equations we obtain for y the mapping

$$(\overline{y} + y - \overline{z})(y + \underline{y} - z) = y^2. \tag{1.4}$$

Equation (1.4) is another special form of a d-P_I which was first obtained in [9]. Thus we have established a quadratic relation, which is, in fact, a degenerate form of a Miura transformation, between two d-P_I's.

In this paper, we shall examine the quadratic relations of continuous and discrete $\mathbb{P}$'s. In the continuous case, we present the quadratic relations for P_{II}, P_{III}, P_V and P_{VI}. The organization we have chosen for the discrete cases is to start with the discrete q-P_{VI} [10] and proceed from this seven-parameter equation down towards equations with fewer parameters.

2. Quadratic relations for continuous Painlevé equations

As we have seen in the introduction, one very simple quadratic relation is that relating the zero-parameter P_{III} to a special case of the full P_{III}. Another, even simpler, case does, however, exist. Let us start with P_{II}

$$u'' = 2u^3 + tu + \alpha \tag{2.1}$$

in which we take $\alpha = 0$. Multiplying by w and introducing $w = u^2$ we obtain the equation

$$w'' = \frac{w'^2}{2w} + 4w^2 + 2tw \tag{2.2}$$

which is equation XX in the Painlevé-Gambier classification.

Another well known quadratic relation is that relating P_V to P_{III} [11]. We start from P_V in the form

$$w'' = w'^2\left(\frac{1}{2w} + \frac{1}{w-1}\right) - \frac{w'}{t} - \frac{(w-1)^2}{t^2}\left(\alpha w + \frac{\beta}{w}\right) + \gamma\frac{w}{t} + \delta\frac{w(w+1)}{w-1} \tag{2.3}$$

and assume that $\alpha = \beta = 0$. Next, we introduce the quadratic dependent variable transformation $w = (u+1)^2/(u-1)^2$ and obtain the P_{III} equation

$$u'' = \frac{u'^2}{u} - \frac{u'}{t} - \frac{\gamma}{4t}(u^2 - 1) - \frac{\delta}{8}\left(u^3 - \frac{1}{u}\right). \tag{2.4}$$

Given this form it is always possible, for non-zero δ, to bring the equation to the form

$$u'' = \frac{u'^2}{u} - \frac{u'}{t} - \frac{a}{t}(u^2 - 1) + u^3 - \frac{1}{u} \tag{2.5}$$

through a scaling of the independent variable. We must point out that although (2.5) contains just one parameter a, it is *not* the one-parameter P_{III} we introduced (and studied exhaustively) in [12]. Equation (2.5) should be understood, rather, as the full P_{III} (1.1) with a constraint on the parameters, namely $\alpha = -\beta$ $(= a)$. On the other hand, if we start from (2.3) with $\delta = 0$, we recover precisely the zero-parameter P_{III}, equation (1.2).

The quadratic relations of P_V and P_{VI} to themselves do not seem to be as well known. In the case of P_V (2.3) one has to make the following remark. If $\delta \neq 0$, the values of α, β and γ can be modified by Schlesinger transformations, so the particular values $\alpha = \beta = 0$ that allow the quadratic transformation to P_{III} are not really special. Another set of particular values for these modifiable parameters is $\beta = -\alpha$, $\gamma = 0$. In that case, a quadratic transformation $v = 4w/(w+1)^2$, i.e. $1 - v = (w-1)^2/(w+1)^2$, and a change of independent variable $z = t^2$ leads, for v as a function of z, to P_V with parameters $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$. The latter are related to the parameters of the initial equation by $\tilde{\alpha} = 0$, $\tilde{\beta} = -4\alpha$, $\tilde{\gamma} = -\delta/4$, $\tilde{\delta} = 0$. The fact that $\tilde{\delta} = 0$ is important, since this value cannot be modified by Schlesinger transformations and this P_V equation is genuinely reduced. As a matter of fact, it has been known for long [2] that this reduced P_V is, in fact, a Miura of the P_{III} equation.

In the case of P_{VI}, we start from

$$w'' = \frac{w'^2}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t}\right) - w'\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{w-t}\right)$$
$$+ \frac{w(w-1)(w-t)}{2t^2(t-1)^2}\left(\alpha - \beta\frac{t}{w^2} + \gamma\frac{t-1}{(w-1)^2} + (1-\delta)\frac{t(t-1)}{(w-t)^2}\right). \tag{2.6}$$

We next assume that $\alpha = \beta = 0$. In this case there exists a quadratic transformation which allows us to convert P_{VI} in the variables w, t again to a P_{VI} for the variables u, s:

$$t = \left(\frac{1+\sqrt{s}}{1-\sqrt{s}}\right)^2 \qquad w = \left(\frac{u+\sqrt{s}}{u-\sqrt{s}}\right)^2. \tag{2.7}$$

If we denote the parameters of the new P_{VI} by $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ the following relations hold:

$$\tilde{\alpha} = \tilde{\beta} = \tfrac{1}{4}\gamma \qquad \tilde{\gamma} = \tilde{\delta} = \tfrac{1}{4}\delta.$$

As we shall see in what follows these relations for continuous $\mathbb{P}$'s have both discrete analogues and discrete consequences.

Our findings on the quadratic relations between Painlevé equations are summarized in table 1.

Table 1.

	Simple singularity	Double singularity
i	P_{II} ($\alpha = 0$)	P_{20}
ii	P_{III} ($\alpha = \beta = 0$)	Zero-parameter P_{III} ($\gamma = \delta = 0$)
iii	P_{III} ($\alpha = -\beta$)	P_V ($\alpha = \beta = 0$)
iv	P_V ($\alpha = -\beta,\ \gamma = 0$)	P_V ($\delta = 0,\ \alpha = 0$)
v	P_{VI} ($\alpha = \beta,\ \gamma = \delta$)	P_{VI} ($\alpha = \beta = 0$)

3. Quadratic relations for discrete Painlevé equations

3.1. The discrete P_{VI} equation

In [10] we have presented the discrete form of the P_{VI} written as the three-point, five-parameter mapping:

$$\frac{(x\overline{x} - z\overline{z})(x\underline{x} - z\underline{z})}{(x\overline{x} - 1)(x\underline{x} - 1)} = \frac{(x-az)(x-z/a)(x-bz)(x-z/b)}{(x-c)(x-1/c)(x-d)(x-1/d)} \tag{3.1}$$

where a, b, c, d, are constants, $z \equiv z_n = z_0\lambda^n$ for some constant λ, $\overline{z} \equiv z_{n+1}$ and $\underline{z} \equiv z_{n-1}$. As we have shown in [13], equation (3.1) possesses an 'asymmetric' form which involves seven parameters. The term 'asymmetric' is used following the QRT [14] terminology and means that the system can be expressed as a system of two two-point mappings. It turns out that imposing quadratic relations on the asymmetric q-P_{VI} forces it back to the symmetric form and thus we shall concentrate on the latter. Exactly the same situation will arise for the equations of subsections 3.2–3.4 below.

Two possibilities of quadratic relations exist: either the right-hand side of (3.1) is a perfect square or it depends only on x^2. Let us explore the first one. The condition for this is $b = a$, $d = c$, in which case we have

$$\frac{(x\overline{x} - z\overline{z})(x\underline{x} - z\underline{z})}{(x\overline{x} - 1)(x\underline{x} - 1)} = \left(\frac{(x-az)(x-z/a)}{(x-c)(x-1/c)}\right)^2. \tag{3.2}$$

Next we introduce the variable y through $y^2 = -(x\overline{x} - z\overline{z})/(x\overline{x} - 1)$ and rewrite (3.2) as the system

$$y\underline{y} = \frac{(x-az)(x-z/a)}{(x-c)(x-1/c)} \qquad x\overline{x} = \frac{y^2 + z\overline{z}}{y^2+1}. \tag{3.3}$$

Equation (3.3) can now be identified as a particular case of the asymmetric q-P_{III} equation, which as was shown by Jimbo and Sakai [15] is a discretization of P_{VI}. Note that here the quantity which is a perfect square is not the initial variable but a more complicated expression. This relation is the discrete analogue of case v of table 1.

The second possibility corresponds to x appearing only as x^2 in the right-hand side of (3.1). The constraint in this case is $b = -a$, $d = -c$. Introducing $X \equiv x^2$, and $Y = (x\overline{x} - z\overline{z})/(x\overline{x} - 1)$ we can rewrite (3.1) as

$$\begin{aligned} Y\underline{Y} &= \frac{(X - a^2z^2)(X - z^2/a^2)}{(X - c^2)(X - 1/c^2)} \\ X\overline{X} &= \left(\frac{Y - z\overline{z}}{Y - 1}\right)^2. \end{aligned} \tag{3.4}$$

Just like system (3.3), equation (3.4) is a subcase of asymmetric q-P_{III}. Since the latter is a discrete P_{VI} this quadratic relation is also the discrete analogue of case v of table 1.

3.2. The discrete, difference, P_V

The three-point mapping form of P_V was presented in [10] and is written as

$$\frac{(x + \overline{x} - z - \overline{z})(x + \underline{x} - z - \underline{z})}{(x + \overline{x})(x + \underline{x})} = \frac{(x - z - a)(x - z + a)(x - z - b)(x - z + b)}{(x - c)(x + c)(x - d)(x + d)} \tag{3.5}$$

where $z \equiv z_n = \alpha n + \beta$. Given the form of (3.5), it is clear that the right-hand side can never depend on x^2 only, so the only possibility for a quadratic relation is when the right-hand side becomes a perfect square. This is true whenever $b = a$, $d = c$. We can, in this case, introduce y through $y^2 = -(x + \overline{x} - z - \overline{z})/(x + \overline{x})$ which results in the system

$$\begin{aligned} y\underline{y} &= \frac{(x - z - a)(x - z + a)}{(x - c)(x + c)} \\ x + \overline{x} &= \frac{z + \overline{z}}{1 + y^2}. \end{aligned} \tag{3.6}$$

This equation is a special form of the discrete P_V we have introduced in [16]. Its general form reads

$$x + \overline{x} = \frac{\tilde{z} + p}{1 - y/t} + \frac{\tilde{z} - p}{1 - ty} \tag{3.7a}$$

$$y\underline{y} = \frac{(x - z)^2 - a^2}{x^2 - c^2} \tag{3.7b}$$

with $\tilde{z} = z + \alpha/2$ and in order to obtain (3.6) we must take $t = i$, $p = 0$. Note that the parameter t cannot be altered through Schlesinger transformations (contrary to a, c and p). In fact, equation (3.7) is a contiguity relation of the solutions of the (continuous) P_{VI} equation and t (or, rather, t^2) is the continuous variable of the latter. The quadratic relation crucially depends on the special value i for t and thus is *not* a contiguity relation valid for arbitrary values of the continuous variable. The constraints for the quadratic relation to exist are incompatible with the continuous limits of (3.5) and (3.6) to P_V.

3.3. The q-discrete P_{V}

The symmetric form of q-P_{V} reads

$$(x\overline{x}-1)(x\underline{x}-1)=\frac{(x-a)(x-1/a)(x-b)(x-1/b)}{(1-zxc)(1-zx/c)} \tag{3.8}$$

with $z=z_0\lambda^n$ for some constant λ. Here again there are two possibilities for a quadratic transformation. First, the right-hand side can be a perfect square. The constraint on the parameters in this case is $b=a$, $c=1$. Introducing the variable y through $y^2=x\overline{x}-1$ we obtain the system

$$\begin{aligned} y\underline{y} &= \frac{(x-a)(x-1/a)}{(1-zx)} \\ x\overline{x} &= y^2+1. \end{aligned} \tag{3.9}$$

The second quadratic relation is obtained when x appears solely in the form of x^2 in the right-hand side of (3.8). This happens whenever $a=-b$, $c=i$. Introducing $Y=x\overline{x}-1$, $X=x^2$ we obtain the system

$$\begin{aligned} Y\underline{Y} &= \frac{(X-a^2)(X-1/a^2)}{(1-z^2X)} \\ X\overline{X} &= (Y+1)^2. \end{aligned} \tag{3.10}$$

Both systems (3.9) and (3.10) are special cases of a system where the second equation is written as $x\overline{x}=(y-p)(y-1/p)$ (or the capitalized version thereof), where p is a free constant. This system is the Miura transform of the asymmetric form of an equation, the symmetric form of which was identified as a q-P_{II} in [9]. The full equation

$$\begin{aligned} \underline{v}v &= \frac{\alpha\zeta w+\gamma\zeta^2}{v(v-1)} \\ w\overline{w} &= \frac{\beta\tilde{\zeta}v+\gamma\tilde{\zeta}^2}{w(w-1)} \end{aligned} \tag{3.11}$$

with $\zeta=\zeta_0\lambda^n$, $\tilde{\zeta}=\zeta\sqrt{\lambda}$ and α,β,γ constants, is, in fact, a q-P_{III} and has been studied in [17]. With the transformation

$$\begin{aligned} x &= \frac{w(v+\alpha\zeta-vw)}{\alpha r\zeta} \\ y &= \frac{vw-\alpha\zeta}{pv} \end{aligned} \tag{3.12}$$

where $a^2=\beta\sqrt{\lambda}/\alpha$, $p^2=\alpha\beta/\gamma\sqrt{\lambda}$ we obtain the generalization of system (3.9)

$$\begin{aligned} y\underline{y} &= \frac{(x-a)(x-1/a)}{(1-zx)} \\ x\overline{x} &= (y-p)(y-1/p) \end{aligned} \tag{3.13}$$

with $z=-a\alpha/\gamma\zeta$. Both quadratic relations obtained above are q-discrete analogues of case iii of table 1.

3.4. The discrete, difference, P_{IV}

Here too, we start with the symmetric form:

$$(x+\overline{x})(x+\underline{x}) = \frac{(x-a)(x+a)(x-b)(x+b)}{(x-z-c)(x-z+c)} \tag{3.14}$$

where $z=\alpha n+\beta$. A quadratic relation exists only if $a=b$, $c=0$. Introducing the variable y through $y^2=-(x+\overline{x})$ we obtain the system

$$y+\underline{y} = \frac{x^2-a^2}{x-z} \tag{3.15a}$$

$$x+\overline{x} = -y^2. \tag{3.15b}$$

This equation is a Miura transform of a particular limiting case of the asymmetric d-P_I we have studied in [18].

$$\begin{aligned} \underline{v}+u+v &= t-\frac{z-a}{u} \\ u+v+\overline{u} &= t-\frac{z+a}{v} \end{aligned} \tag{3.16}$$

with the following relations:

$$x=uv+z \qquad y=v+\overline{u} \tag{3.17}$$

leading to (3.15a) and a generalized form of (3.15b), namely

$$x+\overline{x} = -y(y-t). \tag{3.18}$$

Here t is the (independent) variable of an underlying continuous P_{IV} equation, and the asymmetric d-P_I (3.16) is just the contiguity relation of the solutions of P_{IV}. The limit we refer to is $t=0$, in which case (3.16) corresponds to an asymmetric form of the d-P_0 we examined in the introduction. As in subsection 3.2 above, the quadratic relation, only valid for $t=0$, is not related to a contiguity relation valid for t arbitrary. The quadratic relation obtained here is an essentially discrete property.

3.5. The asymmetric q-P_{III}

In the case of q-P_{III} the quadratic relation exists for the asymmetric case:

$$y\underline{y} = \frac{(x-za)(x-z/a)}{(1-xc)(1-x/c)} \tag{3.19a}$$

$$x\overline{x} = \frac{(y-\tilde{z}c)(y-\tilde{z}/c)}{(1-yd)(1-y/d)} \tag{3.19b}$$

with $z=z_0\lambda^n$, $\tilde{z}=z\sqrt{\lambda}$. The right-hand side of (3.19a) is a perfect square when $a=c=1$, while it is a function of y^2 only if $a=c=i$. In both cases, one finds exactly the relations we have encountered in subsection 3.1 relating this equation to q-P_{VI}. These relations have a historical importance, since it was their derivation that actually led to the discovery of the full q-P_{VI} in the form (3.1) (and also of its asymmetric, seven-parameter generic form).

3.6. The d-P_V, contiguity relation of P_{VI}

In subsection 3.2 we encountered a relation between two discrete P_V equations, the second of which (3.7) is the contiguity relation of P_{VI}. The quadratic relation obtained there was only valid for a fixed value of the independent (continuous) variable of P_{VI}. In what follows we shall examine the quadratic relation of (3.7)

$$x + \overline{x} = \frac{\tilde{z} + p}{1 - y/t} + \frac{\tilde{z} - p}{1 - ty} \tag{3.7a}$$

$$y\underline{y} = \frac{(x-z)^2 - a^2}{x^2 - c^2} \tag{3.7b}$$

(where $z = \alpha n + \beta$) where t is now a free parameter. Clearly, such a relation may exist only if the right-hand side of (3.7b) is a perfect square, i.e. $a = c = 0$. In this case we introduce $y = Y^2$ and find $Y\underline{Y} = 1 - z/x$ which can be solved for x, leading to the equation

$$\frac{\overline{z}}{1 - Y\overline{Y}} + \frac{z}{1 - Y\underline{Y}} = \frac{\tilde{z} + p}{1 - Y^2/t} + \frac{\tilde{z} - p}{1 - tY^2}. \tag{3.20}$$

This equation can further be brought to canonical form through the transformation $t = r^2$, $Y = (1 - rX)/(r - X)$. We thus find

$$\frac{\overline{z}}{1 - X\overline{X}} + \frac{z}{1 - X\underline{X}} = \frac{p}{2} + \tilde{z} + \frac{p(s^2 - 1)X + s(1 - X^2)\tilde{z}}{2(X + s)(1 + Xs)} \tag{3.21}$$

where $s = -(1 + r^2)/2r$. Equation (3.21) is a special case of the equation introduced in [19] (under the name 'master d-P_{II}'). As we have shown there, equation (3.21) is also a contiguity relation of solutions of P_{VI} (although X as it enters in (3.21) is not the variable of P_{VI}, but rather is related to it through a homographic transformation). Thus the quadratic relation between (3.7) and (3.21) is nothing but a consequence of the quadratic relation we obtained in section 2 for P_{VI}, i.e. case v of table 1. Moreover, this relation has an even more interesting interpretation. Equation (3.21) has P_{III} as its continuous limit. Thus the quadratic relation between (3.7) (which is a discrete P_V) and (3.7) is the *discrete analogue* of the relation between P_V and P_{III}, i.e. of case iii of table 1. This is the first time, to our knowledge, that such a relation has been obtained explicitly in a discrete setting.

The general asymmetric form of the 'master d-P_{II}' equation is

$$\frac{\overline{z}}{1 - X\overline{X}} + \frac{z}{1 - X\underline{X}} = a + \tilde{z} + \frac{b(s^2 - 1)X + s(1 - X^2)(\tilde{z}/2 + (-1)^n c)}{(X + s)(1 + Xs)}. \tag{3.22}$$

A question that seems natural at this point is whether (3.23) has any quadratic relations apart from the one we introduced in the previous paragraph. We find readily that the right-hand side of (3.22) depends only on X^2 if $s = i, b = 0$. Note that since s is related to the continuous variable of the P_{VI}, the solutions of which have (3.22) as a contiguity relation, the quadratic relation we are going to obtain exists only for a fixed value of this independent variable. In this case we introduce $x = X^2$, $y = z/(1 - X\underline{X})$ and find

$$\overline{y} + y = a + \tilde{z} + \frac{1 - x}{1 + x}(\tilde{z}/2 + (-1)^n c) \tag{3.23a}$$

$$x\underline{x} = \left(\frac{z - y}{y}\right)^2. \tag{3.23b}$$

We can now solve (3.23*a*) for x and obtain a single equation for y. Finally, we find

$$\left(\frac{\overline{y}+y-a-3\tilde{z}/2-(-1)^n c}{\overline{y}+y-a-\tilde{z}/2+(-1)^n c}\right)\left(\frac{y+\underline{y}-a-3\underset{\sim}{z}/2+(-1)^n c}{\overline{y}+\underline{y}-a-\underset{\sim}{z}/2-(-1)^n c}\right)=\left(\frac{z-y}{y}\right)^2. \tag{3.24}$$

Equation (3.24) belongs to the (asymmetric) d-P_V family (3.5) with only two factors instead of four in both the numerator and the denominator of the right-hand side. An equation of the form (3.24) was obtained in [13] where we have shown that it introduces a Miura transformation for (asymmetric) d-P_V (3.5). A translation of y suffices in order to bring (3.24) into canonical form.

3.7. *The asymmetric d-P_{II}, contiguity relation of P_V*

The general form of the asymmetric d-P_{II} is

$$y+\underline{y}=\frac{zx+a}{x^2-1} \tag{3.25a}$$

$$x+\overline{x}=\frac{\tilde{z}y+b}{y^2-1} \tag{3.25b}$$

where $z=\alpha n+\beta$ and $\tilde{z}=\zeta+\alpha/2$. A first, obvious, quadratic relation can be obtained when both a and b vanish. In that case the system (3.24) can be rewritten as a symmetric mapping for a single function:

$$\underline{v}+\overline{v}=\frac{zv}{v^2-1}. \tag{3.26}$$

Multiplying both sides of (3.26) by v and introducing $X=v^2$, $W=v\overline{v}$, we find this system:

$$W+\underline{W}=\frac{zX}{X-1} \tag{3.27a}$$

$$X\overline{X}=W^2. \tag{3.27b}$$

If one wants, one can solve (3.27*a*) for X to find an equation for W only, namely

$$\frac{(W+\overline{W}-\overline{z})(W+\underline{W}-z)}{(W+\overline{W})(W+\underline{W})}=\frac{1}{W^2}. \tag{3.28}$$

This equation is a particular case of an equation already identified in [20] as the Miura transform of the equation known as the 'alternate d-P_{II}' equation:

$$\frac{\overline{z}}{1+u\overline{u}}+\frac{z}{1+u\underline{u}}=u-\frac{1}{u}+z+\mu. \tag{3.29}$$

The Miura transformation is $X=z\underline{u}/(1+u\underline{u})+1$, $W=uX+\mu/2$. From (3.29) it follows that $W=\overline{X}/u-\mu/2$, so in general the relation between X, $\overline{X}$ and W should be written as

$$X\overline{X}=W^2-\mu^2/4. \tag{3.30}$$

Moreover, equation (3.27*a*) follows immediately from using the first expression of W and the downshift of the second one for $\underline{W}$. So it is the system (3.27*a*) and (3.30) which is the complete Miura transform of (3.29), and the equation for W alone is obtained in replacing W^2 by $W^2-\mu^2/4$ in (3.28). Still, equation (3.27) is a particular case of it, when the constraint $\mu=0$ holds. Since (3.29) is known to be the contiguity relation of solutions of P_{III} [21], while (3.25) is the contiguity relation of solutions of P_V one could naively think that this quadratic relation is a consequence of the relation between P_V and P_{III}, i.e. case iii of table 1. However, this is not the case. Indeed, the quadratic transformation does not lead to the solution u of P_{III}, but to its Miura transform X which satisfies the reduced P_V with $\delta = 0$. Thus this relation is, in fact, a discrete consequence of the quadratic relation between P_V and its reduced form, i.e. case iv of table 1. Moreover, it is the discrete analogue of case i of table 1, since in the continuous limit (3.26) and (3.28) go, respectively, to P_{II} ($\alpha = 0$) and P_{20}.

The consequence of the relation between P_V and P_{III}, i.e. case iii of table 1, can also be seen as a quadratic relation between the asymmetric d-P_{II} and the alternate d-P_{II} as follows. Let us start from the latter, equation (3.29) with the constraint that $\mu = (\overline{z} - z)/2$. Then if we define

$$
\begin{aligned}
w &= \frac{1+u^2}{2u} \\
x &= \frac{u - \underline{u}}{1 - u\underline{u}} \\
y &= \frac{z}{2(x-w)} - 1
\end{aligned}
\tag{3.31}
$$

one finds that x and y satisfy (3.25) with $a = 0$, $b = (\overline{z} - z)/2$. This does not look so much like a quadratic relation unless one realizes that

$$
\frac{w-1}{w+1} = \left(\frac{u-1}{u+1}\right)^2 \tag{3.32}
$$

so, in fact, equation (3.32) is indeed a quadratic relation between the solution u of P_{III} and the object $(w-1)/(w+1)$. In fact, this relation is exactly of the form expected from section 2 which means that the quantity $(w-1)/(w+1)$ is itself a solution of P_V, even though w *seems* to be only an object related through a Miura transform to x and y that satisfy asymmetric d-P_{II} and therefore P_V (more precisely, the quantities that satisfy P_V in canonical form are $(x-1)/(x+1)$, $(y-1)/(y+1)$).

3.8. The asymmetric d-P_I, contiguity relation of P_{IV}

We have already seen in subsection 3.4 that a Miura transform of this equation, namely (3.15*a*)–(3.18) was related through a quadratic transformation to the discrete, difference, P_{IV}. Moreover, in the introduction we have already considered the symmetric form of this equation. For completeness, we now consider the fully asymmetric d-P_I which we do not write as a two-variable system (3.16) but as a single equation, with an explicit $(-1)^n$ contribution

$$
\overline{x} + x + \underline{x} = t - \frac{z + (-1)^n a}{x} \tag{3.33}
$$

with $z = \alpha n + \beta$. With the same notation as in the introduction, $X = x^2$, $y = x\overline{x}$, and provided $t = 0$, we find the generalized form of (1.4), namely

$$
(\overline{y} + y - \overline{z} + (-1)^{n+1} a)(y + \underline{y} - z + (-1)^n a) = y^2. \tag{3.34}
$$

This equation has been identified in [13] as a Miura of, again, the discrete, difference, d-P_{IV} but in its asymmetric form (for particular values of the parameters of the latter). This quadratic relation is an essentially discrete one.

We have seen in subsection 3.7 that there are two ways for the alternate d-P_{II} (3.29), contiguity relation of P_{III}, to be related to the asymmetric d-P_{II} (3.25) and we did not find any more quadratic relations for it, nor did we find any for the alternate d-P_I equation,

$$\frac{\overline{z}}{x+\overline{x}} + \frac{z}{x+\underline{x}} = x^2 + t \tag{3.35}$$

which is the contiguity relation of P_{II}. This completes our survey of quadratic relations for discrete Painlevé equations.

4. Conclusion

In this paper we have examined the quadratic relations that exist between the solutions of Painlevé equations. These quadratic relations can be considered as degenerate Miura transformations and exist only for some of the $\mathbb{P}$'s when special constraints are satisfied.

The discrete Painlevé equations also possess quadratic relations. Some of them are just the discrete analogues of the relations for the continuous $\mathbb{P}$'s, while others are their direct consequences (and sometimes both at the same time). Prominent among these quadratic relations is the discrete analogue of the between P_V and P_{III} which has been missing until now and which is also the discrete consequence of the quadratic relation between solutions of P_{VI}. One must also stress the historical role played by the quadratic relations in the discovery of the discrete q-P_{VI} equation.

All in all, the study of the quadratic relations turned out to be particularly interesting and revealed another (not so well known) aspect of the Painlevé equations, thus confirming the wealth of their structure.

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